Matched differential calculus on the quantum groups $GL_q(2,C), SL_q(2,C), C_q(2|0)$.

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We proposed the construction of the differential calculus on the quantum group and its subgroup with the property of the natural reduction: the differential calculus on the quantum group $GL_q(2,C)$ has to contain the differential calculus on the quantum subgroup $SL_q(2,C)$ and quantum plane $C_q(2|0)$ ("quantum matrjoshka"). We found, that there are two differential calculi, associated to the left differential Maurer–Cartan 1-forms and to the right differential 1-forms. Matched reduction take the degeneracy between the left and right differentials. The classical limit ($q \to 1$) of the "left" differential calculus and of the "right" differential calculus is undeformed differential calculus. The condition $\mathcal{D}_q G = 1$ gives the differential calculus on $SL_q(2,C)$, which contains the differential calculus on the quantum plane $C_q(2|0)$.

1 Introduction

In this paper we discuss a new effect appearing in the differential calculus on quantum groups and its subgroups.

After the discovery [1,2] q-deformed or quantum groups a peculiar attention was paid to constructing bicovariant differential calculus on these quantum groups [3].But bicovariant differential calculus on $GL_q(N)$ cannot be expressed in terms of differential calculus on $SL_q(N)$, because the Maurer–Cartan 1-forms have q^2-commutation relations with quantum determinant $\mathcal{D}_q$. Such an approach was used for quantum groups in [4-8] with undeformed classical Leibnitz rule for the exterior derivative. Another approach can be formulated following the Faddeev–Pjatov idea that the exterior derivative obeys the modified version of the Leibnitz rule [9].

We are going to describe another approach with main assumptions: 1) undeformed classical Leibnitz rule, 2) the differential calculus on the quantum group $GL_q(2,C)$ has to contain the differential calculus on the quantum subgroup $SL_q(2,C)$. As the result, we will find that there are two differential calculus on the quantum group $GL_q(2,C)$, associated to the left differential forms and to the right differential forms. The classical limit ($q \to 1$) of the "left" differential calculus and the "right" differential calculus is the undeformed differential calculus.

The condition $\det_q G = 1$ gives the differential calculus on $SL_q(2,C)$. If the parameter b or c is equal to zero, we will find the differential calculus on the quantum plane $C_q(2|0)$ [6].

Let us briefly discuss the content of the paper. In the second section the basic notations of the differential calculus on the quantum group are introduced. In the third section we will be dealing with the left 1-forms θ and the left differential δ_L. We find the commutation relations for the left 1-forms and the parameters and we describe the quantum algebras for the vector fields for $GL_q(2,C)$, but only one of them has the Drinfeld–Jimbo form. This choice fixes the form of the quantum trace $\mathrm{Tr}_q\theta$. We derive formulas corresponding to the commutation relations between the parameters and its differentials for $GL_q(2,C)$, which contains $SL_q(2,C)$ case and $C_q(2|0)$ case. In section 4 we considered the $SL_q(2,C)$ case which also containts $C_q(2|0)$ case. In section 5 we introduce the right 1-forms ω and right differential $\overrightarrow{\delta}_R$ like in the previous sections and recovered the results [10].

2 Differential calculus on quantum group $GL_q(2,C)$.

We first recall some basic notations about differential calculus on quantum group. The starting point for our consideration is the Hopf algebras $Fun\,(GL_q(2,C))$.

Comultiplication, counit and antipode is determined by

$$\Delta\left(T^i_k\right) = T^i_l \otimes T^l_k \tag{2.1}$$

$$e\left(T^i_k\right) = \delta^i_k \tag{2.2}$$

$$S\left(T^i_k\right) = \left(T^{-1}\right)^i_k \tag{2.3}$$

Non-commuting matrix entries $T = \begin{pmatrix} T^1_1 & T^1_2 \\ T^1_1 & T^2_2 \end{pmatrix}$ satisfies the RTT-relation

$$R^{ij}_{kl} T^k_m T^l_n = T^j_l T^i_k R^{kl}_{mn} \tag{2.4}$$

and the quantum Yang-Baxter equation

$$R^{i_1 j_1}_{i_2 j_2} R^{i_2 k_1}_{i_3 k_2} R^{j_2 k_2}_{j_3 k_3} = R^{j_1 k_1}_{j_2 k_2} R^{i_1 k_2}_{i_2 k_3} R^{i_2 j_2}_{i_3 j_3} \tag{2.5}$$

where

$$R^{ij}_{kl} = \begin{pmatrix} R^{11}_{11} & R^{11}_{12} & R^{11}_{21} & R^{11}_{22} \\ R^{12}_{11} & R^{12}_{12} & R^{12}_{21} & R^{12}_{22} \\ R^{21}_{11} & R^{21}_{12} & R^{21}_{21} & R^{21}_{22} \\ R^{22}_{11} & R^{22}_{12} & R^{22}_{21} & R^{22}_{22} \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{2.6}$$

where the rows and columns are numerated in the order 11, 12, 21, 22, or equivalently as composite indices 1,2,3,4 and $\lambda = q - q^{-1}$.

An important element is the quantum determinant

$$\mathcal{D}_q = \det{}_q T = \sum_{i,k=1}^{2} (-q)^{I(ik)} T^1_i T^2_k \tag{2.7}$$

where $I(ik)$ is the number of inversions, or through the q-deformed Levi-Civita tensor

$$\epsilon_q^{ik} = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$$

$$\epsilon_q \mathcal{D}_q = T^+ \epsilon_q T = T \epsilon_q T^+ \tag{2.8}$$

The coproduct Δ, counit e and antipode S on $\mathcal{D}_q$ defined by

$$\Delta(\mathcal{D}_q) = \mathcal{D}_q \otimes \mathcal{D}_q, \quad e(\mathcal{D}_q) = 1, \quad S(\mathcal{D}_q) = \mathcal{D}_q^{-1} \tag{2.9}$$

Adding $\mathcal{D}_q^{-1}$ to the algebra, we suppose that $\mathcal{D}_q \neq 0$. Using (2.4,2.5) to obtain the commutation relations between group elements

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix} \tag{2.10}$$

it can obtain the following component relations [3]

$$\begin{array}{lll} ab = q\ ba & bc = cb & cd = q\ dc \\ ac = q\ ca & bd = q\ db & \\ ad = da + \lambda\ bc, & & \end{array} \tag{2.11}$$

The quantum determinant $\mathcal{D}_q$ is central element and commutes with all group elements

$$T_k^i \mathcal{D}_q = \mathcal{D}_q T_k^i \tag{2.12}$$

One can introduce the left-hand side exterior derivative

$$\delta_L f = \left(f \frac{\overleftarrow{\partial}}{\partial T_k^i} \right) \delta_L T_k^i \tag{2.13}$$

satisfying 1) Leibnitz rule

$$\delta_L(f\theta) = (f\theta) \overleftarrow{\delta}_L = f(\delta_L\theta) + (-1)^{\eta(\theta)}(\delta_L f)\theta \tag{2.14}$$

and 2) Poincare lemma

$$\delta_L^2 f = 0 \tag{2.15}$$

where $\eta(\theta)$ is the degree of differentional form θ and f is arbitrary function of the group elements.

We wish to construct a quantum algebra of the left vector fields on the quantum group $GL_q(2, C)$. To do this we need to introduce an infinitely small neighborhood $\delta_L T$ of the unity of the group, and we need to determine the commutation relations between the parameters of the group and its differentials. Equivalently, we need to determine the

commutation relations between the parameters of the group and the Maurer-Cartan left 1-forms, which are given by

$$\theta = S(T)\delta_L T = \begin{pmatrix} \theta^1 & \theta^2 \\ \theta^3 & \theta^4 \end{pmatrix} = \begin{pmatrix} \theta^1_1 & \theta^1_2 \\ \theta^2_1 & \theta^2_2 \end{pmatrix} \tag{2.16}$$

where

$$\begin{aligned} \theta^1 &= \mathcal{D}_q^{-1}(d\delta_L a - q^{-1} b\delta_L c) = \theta^1_1 \qquad & \theta^3 &= D_q^{-1}(a\delta_L c - qc\delta_L a) = \theta^2_1 \\ \theta^2 &= \mathcal{D}_q^{-1}(d\delta_L b - q^{-1} b\delta_L d) = \theta^1_2 \qquad & \theta^4 &= D_q^{-1}(a\delta_L d - qc\delta_L b) = \theta^2_2 \end{aligned} \tag{2.17}$$

Left coaction Δ_L extends to Maurer-Cartan left 1-forms such that

$$\Delta_L \circ \delta_L = (id \otimes \delta_L) \circ \Delta$$

$$\Delta_L(\delta_L T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \delta a & \delta b \\ \delta c & \delta d \end{pmatrix} \tag{2.18}$$

$$\Delta_L(\theta(\delta_L)) = 1 \otimes \theta(\delta_L)$$

In the same way we define the right-hand side exterior derivative

$$\overrightarrow{\delta_R}\, f = \delta_R \overrightarrow{T^k_i}\ \frac{\overrightarrow{\partial}}{\partial T^k_i} f \tag{2.19}$$

satisfying the classical Leibnitz rule

$$\overrightarrow{\delta_R}\,(\omega g) = (\overrightarrow{\delta_R}\,\omega)g + (-1)^{\eta(\omega)}\omega(\overrightarrow{\delta_R}\,g) \tag{2.20}$$

and Poincare lemma $\overrightarrow{\delta_R}^{\,2} f = 0$.

Let us define the Maurer-Cartan right-invariant 1-form ω on T

$$\omega = \delta_R T\ S(T) = \begin{pmatrix} \omega^1 & \omega^2 \\ \omega^3 & \omega^4 \end{pmatrix} = \begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix}, \tag{2.21}$$

where

$$\begin{aligned} \omega^1 &= (\overrightarrow{\delta_R} a\ d - q\ \ \overrightarrow{\delta_R} b\ c)\mathcal{D}_q^{-1} = \omega^2_1, \qquad & \omega^3 &= (\overrightarrow{\delta_R} c\ d - q\ \ \overrightarrow{\delta_R} d\ c)\mathcal{D}_q^{-1} = \omega^2_1, \\ \omega^2 &= (\overrightarrow{\delta_R} b\ a - \tfrac{1}{q}\ \ \overrightarrow{\delta_R} a\ b\ )\mathcal{D}_q^{-1} = \omega^1_2, \qquad & \omega^4 &= (\overrightarrow{\delta_R} d\ a - \tfrac{1}{q}\ \overrightarrow{\delta_R} c\ b)\mathcal{D}_q^{-1} = \omega^2_2, \end{aligned} \tag{2.22}$$

We note, that left differential arranges after left derivatives, but right differential before right derivatives.

Right coaction $\Delta_R(\vec{\delta_R}) = \vec{\delta_R} \otimes 1$:

$$\Delta_R(\vec{\delta_R}\, T) = \begin{pmatrix} \vec{\delta_R}\, a & \vec{\delta_R}\, b \\ \vec{\delta_R}\, c & \vec{\delta_R}\, d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.23}$$

$$\Delta_R(\omega(\vec{\delta_R})) = \omega \otimes 1$$

Notice, that on the group parameters T^i_k the action Δ_R and Δ_L coincides with the action Δ.

To specify a differential calculus it is necessary to define commutation relations between T^i_k and their differentials δT^i_k.

Throughout the recent development of differential calculus on quantum groups and quantum spaces, two principal concepts are readily seen. First of them, formulated by Woronowicz [5], is known as bicovariant differential calculus on quantum groups:

$$(1 \otimes \Delta_R)\Delta_L = (\Delta_L \otimes 1)\Delta_R \tag{2.24}$$

The examples were considered by Schupp, Wats, Zumino [11], Jurčo[8], Sudberry [12], Müller-Hoissen [13,14], Isaev, Pjatov [15], Demidov [16], Manin [7] and others [17–20]. But the natural way of obtaining the $SL_q(N)$-differential calculus by performing reduction from the $GL_q(N)$-calculus ($\mathcal{D}_q = 1$) failed in the quantum case.

Another concept, introduced by Woronowicz [4] and Schirrmacher, Wess, Zumino [10] proceeds from the requirement of left- (right-) invariant differential calculus only. We will consider the last concept.

In this paper we propose the construction of the differential calculus on the group and its subgroup with the property of ”quantum matrjoshka” (matched reduction): the differential calculus on the quantum group $GL_q(2, C)$ has to contain the differential calculus on the quantum subgroup $SL_q(2, C)$ and quantum plane $C_q(2|0)$.

We postulate, that the quantum determinant $\mathcal{D}_q$ is a central element for θ or ω 1-forms

$$\theta^i \mathcal{D}_q = \mathcal{D}_q \theta^i, D_q\omega = \omega D_q \tag{2.25}$$

in contrast to [11,16,18], where $\theta^i \mathcal{D}_q = q^{-2}\mathcal{D}_q\theta^i$ for the bicovariant differential calculus. A consequence of these conditions is that we can take $\mathcal{D}_q = \det_q T = 1$ for $SL_q(2, C)$ and will obtain the matched differential calcules on the $GL_q(2, C)$ and the $SL_q(2, C)$.

3 Left differentioal calculus on the $GL_q(2, C)$.

Let us first consider the case of the left invariant differential calculi on the $GL_q(2, C)$. Matched differential calculus has to satisfy seven basic conditions, six of them are the conditions of consistency with RTT relations and last condition is the condition of consistency of differential calculus on the quantum group and its subgroup.

I) Commutation relations between the T^i_k and their differentials can be expressed in terms of the Maurer–Cartan left 1-forms[13]

$$\theta^\alpha_\beta T^i_k = T^i_m \left(\bar{A}^m_k\right)^{\alpha\gamma}_{\delta\beta} \theta^\delta_\gamma \tag{3.1}$$

where $\bar{A}^m_k = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and A, B, C, D are 4×4 matrices with complex entries.

II) The requirement, that θ^α_β commute with commutation relations (2.4)

$$\theta^i_j \left(R^{\alpha n}_{\beta m} T^\beta_\delta T^m_k - T^n_m T^\alpha_\beta R^{\beta m}_{\delta k} \right) = 0 \tag{3.2}$$

leads to some additional equations for the matrices A, B, C, D

$$R^{\alpha n}_{\beta m} \ \bar{A}^\beta_\delta \bar{A}^m_k = \bar{A}^n_m \bar{A}^\alpha_\beta \ R^{\beta m}_{\delta k} \tag{3.3}$$

A, B, C, D have to form a representation of a, b, c, d

$$\begin{array}{c} AB = q\ BA \quad AC = q\ CA \quad BC = CB \\ BD = q\ DB \quad CD = q\ DC \\ AD = DA + (q - \frac{1}{q})BC; \end{array} \tag{3.4}$$

From the definition of the left θ-form, as $\theta = S(T^{-1})\delta_L T$ we find

$$\delta_L T^\beta_\alpha = T^\gamma_\alpha \theta^\beta_\gamma \tag{3.5}$$

and now we can use left exterior derivative δ_L.

III) For the quantum determinant $\mathcal{D}_q$ we define quantum trace by the definition

$$\delta_L \mathcal{D}_q = \mathcal{D}_q \mathrm{Tr}_q \theta \tag{3.6}$$

Using (3.1) and (3.5) we obtain

$$\begin{aligned} \delta_L(ad - q\ bc) &= \mathcal{D}_q \left(\theta^1 + A^4_l \theta^l - \tfrac{1}{q} B^3_l \theta^l \right) = \mathcal{D}_q \left(\theta^4 + D^1_l \theta^l - q C^2_l \theta^l \right) \\ \delta_L(ad - q\ cb) &= \mathcal{D}_q \left(\theta^4 + D^1_l \theta^l - q B^3_l \theta^l \right) \\ \delta_L(da - \tfrac{1}{q}\ bc) &= \mathcal{D}_q \left(\theta^1 + A^4_l \theta^l - \tfrac{1}{q} C^2_l \theta^l \right) \\ \delta_L(da - \tfrac{1}{q}\ cb) &= \mathcal{D}_q \left(\theta^1 + A^4_l \theta^l - \tfrac{1}{q} B^3_l \theta^l \right) = \mathcal{D}_q \left(\theta^4 + D^1_l \theta^l - q C^2_l \theta^l \right) \end{aligned} \tag{3.7}$$

and additional equations

$$B_l^3 = C_l^2$$

$$\theta^2 + B_l^1\theta^l - qA_l^2\theta^l = 0 \tag{3.8}$$

$$\theta^3 + C_l^4\theta^l - \tfrac{1}{q}D_l^3\theta^l = 0$$

IV) Using the Maurer–Cartan equations $\delta\theta = -\theta\wedge\theta$ we find the conditions of consistency with RTT relations

$$\delta_L(R_{12}T_1T_2 - T_2T_1R_{12}) = 0 \tag{3.9}$$

as a consequence we obtain the same equations as (2.27–2.28).

V) Using (3.2) we receive

$$\theta_k^i\mathcal{D}_q = \mathcal{D}_q\left(AD - qBC\right)_m^i\theta_k^m$$

and as a consequence we obtain (3.4).

VI) Differentiating commutation relations (3.1) and using the Maurer–Cartan equation we find some additional equations for A, B, C, D

$$\delta_L\theta_\beta^\alpha T_k^n - \theta_\beta^\alpha\delta_L T_k^n = \delta_L T_m^n\left(\bar{A}_k^m\right)_{\delta\beta}^{\alpha\gamma}\theta_\gamma^\delta + T_m^n\left(\bar{A}_k^m\right)_{\delta\beta}^{\alpha\gamma}\delta_L\theta_\gamma^\delta - \theta_\gamma^\alpha\theta_\beta^\gamma T_k^n - \theta_\beta^\alpha T_\gamma^n\theta_k^\gamma =$$

$$= T_\rho^n\theta_m^\rho\left(\bar{A}_k^m\right)_{\delta\beta}^{\alpha\gamma}\theta_\gamma^\delta - T_m^n\left(\bar{A}_k^m\right)_{\delta\beta}^{\alpha\gamma}\theta_\rho^\delta\theta_\gamma^{rho}$$

or

$$T_f^n\left[\left(\bar{A}_m^f\right)_{\tau\gamma}^{\alpha\varphi}\left(\bar{A}_k^m\right)_{\sigma\beta}^{\gamma\beta} + \left(\bar{A}_\sigma^f\right)_{\tau\beta}^{\alpha\varphi}\delta_k^\rho + \left(\bar{A}_k^\varphi\right)_{\gamma\beta}^{\alpha\rho}\delta_\tau^f - \left(\bar{A}_k^f\right)_{\tau\beta}^{\alpha\rho}\delta_\varphi^\sigma\right]\theta_\varphi^\tau\theta_\rho^\sigma = 0$$

We have q-deformation of algebra of θ-forms:

$$\left(\bar{A}_m^f\right)_{\tau\gamma}^{\alpha\varphi}\left(\bar{A}_k^m\right)_{\sigma\beta}^{\gamma\rho}\theta_\varphi^\tau\theta_\rho^\sigma + \left(\bar{A}_\sigma^f\right)_{\tau\beta}^{\alpha\varphi}\theta_\varphi^\tau\theta_k^\sigma + \left(\bar{A}_k^\varphi\right)_{\sigma\beta}^{\alpha\rho}\theta_\varphi^f\theta_\rho^\sigma - \left(\bar{A}_k^f\right)_{\tau\beta}^{\alpha\rho}\theta_\sigma^\tau\theta_\rho^\sigma = 0 \tag{3.10}$$

VII) The last condition is the basic condition which differs matched differential calculus from the bicovariant differential calculus

$$\theta^k\mathcal{D}_q = \mathcal{D}_q\theta^k$$

which leads to the equation $AD - qBC = 1$.

To solve the system of equations for the matrix $\bar{A}$ following the conditions (I–VII) we will consider the representation for entries of $\bar{A}$ with

$$B = C = 0 \qquad AD = 1$$

From (3.7) we have obtained only two different expressions

$$\delta_L\mathcal{D}_q = \mathcal{D}_q\left[D_1^1\theta^1 + (1 + D_4^1)\theta^4\right], \tag{3.11}$$

$$\delta_L \mathcal{D}_q = \mathcal{D}_q \left[(1 + A^4_1)\theta^1 + A^4_4 \theta^4 \right],$$

where we used composite indeces 1,2,3,4.

Finally we find matrices A, D in terms of independent variables $\beta = D^1_1,\ \alpha = A^4_4$

$$A = \begin{pmatrix} 1-\alpha+\frac{\alpha}{\beta} & 0 & 0 & \frac{(1-\alpha)\alpha}{\beta} \\ 0 & \frac{1}{q} & 0 & 0 \\ 0 & 0 & \frac{1}{q} & 0 \\ \beta - 1 & 0 & 0 & \alpha \end{pmatrix}, \quad D = \begin{pmatrix} \beta & 0 & 0 & \alpha - 1 \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ \frac{(1-\beta)\beta}{\alpha} & 0 & 0 & 1-\beta+\frac{\beta}{\alpha} \end{pmatrix} \tag{3.12}$$

From the commutation relations for θ-forms for given A and D we obtain the following commutation relations

$$\theta^1\theta^4 + \theta^4\theta^1 = -\frac{(\alpha-1)}{\beta}\left(\theta^4\right)^2 - \frac{(\beta-1)}{\alpha}\left(\theta^1\right)^2 - \frac{-\beta+q^2\alpha}{\beta\alpha}\theta^2\theta^3 = \tag{3.13}$$

$$= -\frac{(\beta+\alpha-\beta\alpha)}{(1-\beta)\beta}\left(\theta^4\right)^2 - \frac{(\beta-\alpha-\beta^2)}{\beta+2\alpha-\beta\alpha}\left(\theta^1\right)^2 - \frac{(-\beta+q^2\alpha)(1-q^2-\alpha)\alpha}{(\beta+2\alpha-\beta\alpha)(1-\beta)\beta}\theta^2\theta^3 =$$

$$= -\frac{(-\beta+\alpha-\alpha^2)}{\alpha+2\beta-\beta\alpha}\left(\theta^4\right)^2 - \frac{(\beta+\alpha-\beta\alpha)}{(1-\alpha)\alpha}\left(\theta^1\right)^2 - \frac{(-\beta+q^2\alpha)(1-q^{-2}-\alpha)\beta}{(1-\beta)(\alpha+2\beta-\beta\alpha)\alpha}\theta^2\theta^3$$

$$(\theta^2)^2 = (\theta^3)^2 = 0, \qquad \theta^2\theta^3 + q^2\theta^3\theta^2 = 0 \tag{3.14}$$

which are compatible, when

$$\left(\theta^1\right)^2 = 0; \ \left(\theta^4\right)^2 = 0; \ \beta = q^2\alpha \tag{3.15}$$

Using (3.15), we have

$$\theta^1\theta^4 + \theta^4\theta^1 = 0 \tag{3.16}$$

The remaining commutation relations are

$$\begin{aligned} q^2\alpha\theta^2\theta^1 + (q^{-2}+1-\alpha)\theta^1\theta^2 + (\alpha-1)\theta^2\theta^4 + q^{-2}(1-\alpha)\theta^4\theta^2 &= 0 \\ q^2(1-q^2\alpha)\theta^2\theta^1 + (q^2\alpha-1)\theta^1\theta^2 + (1+q^2-q^2\alpha)\theta^2\theta^4 + \alpha\theta^4\theta^2 &= 0 \\ q^2\alpha\theta^1\theta^3 + (q^{-2}+1-\alpha)\theta^3\theta^1 + (\alpha-1)\theta^4\theta^3 + q^{-2}(1-\alpha)\theta^3\theta^4 &= 0 \\ q^2(1-q^2\alpha)\theta^1\theta^3 + (q^2\alpha-1)\theta^3\theta^1 + (1+q^2-q^2\alpha)\theta^4\theta^3 + \alpha\theta^3\theta^4 &= 0 \end{aligned} \tag{3.17}$$

In fact the parameter α plays the role of the definition of the q-trace [3,18,21]

$$\delta_L \mathcal{D}_q = \alpha \mathcal{D}_q (q^2\theta^1 + \theta^4)$$

Thus we obtained the one–parameter family differential calculus on the $GL_q(2,C)$.

The next step is to construct a left vector field algebra for the quantum group $GL_q(2,C)$. By definition, the effect of applying the left differential to an arbitrary function on the quantum group is

$$\delta_L f = \frac{\overleftarrow{\partial} f}{\partial T^i_k} \delta_L T^i_k = \left(f \overleftarrow{\nabla}_k \right) \theta^k \tag{3.18}$$

where $\overleftarrow{\nabla}_k$ are the left vector fields on the quantum group.

From the lemma Poincare

$$\delta_L^2 f = -(f \overleftarrow{\nabla}_k) \delta_L \theta^k - (f \overleftarrow{\nabla}_k)(\overleftarrow{\nabla}_l \theta^l)\theta^k = 0$$

and the Maurer–Cartan equations we find the algebra of the left vector fields for an arbitrary α

$$\begin{gathered}
\overleftarrow{\nabla}_3 \overleftarrow{\nabla}_2 - q^2 \overleftarrow{\nabla}_2 \overleftarrow{\nabla}_3 = \hat{\overleftarrow{\nabla}}_1, \\
q^2 \overleftarrow{\nabla}_2 \hat{\overleftarrow{\nabla}}_1 - q^{-2} \hat{\overleftarrow{\nabla}}_1 \overleftarrow{\nabla}_2 = (1+q^2) \overleftarrow{\nabla}_2 \\
q^2 \overleftarrow{\nabla}_1 \overleftarrow{\nabla}_3 - q^{-2} \overleftarrow{\nabla}_3 \overleftarrow{\nabla}_1 = (1+q^2) \overleftarrow{\nabla}_3 \\
\left[\overleftarrow{\nabla}_4, \overleftarrow{\nabla}_2 \right] = -(1+q^2)(1-\alpha) \overleftarrow{\nabla}_2 \hat{\overleftarrow{\nabla}}_1 + [q + (1+q^2)(1-\alpha)] \overleftarrow{\nabla}_2 \\
\left[\overleftarrow{\nabla}_3, \overleftarrow{\nabla}_4 \right] = -(1+q^2)(1-\alpha) \hat{\overleftarrow{\nabla}}_1 \overleftarrow{\nabla}_3 + [1 + (1+q^2)(1-\alpha)] \overleftarrow{\nabla}_3 \\
\left[\overleftarrow{\nabla}_4, \hat{\overleftarrow{\nabla}}_1 \right] = 0
\end{gathered} \tag{3.19}$$

where $\hat{\overleftarrow{\nabla}}_1$ represents the combination

$$\hat{\overleftarrow{\nabla}}_1 = \overleftarrow{\nabla}_1 - q^2 \overleftarrow{\nabla}_4 \tag{3.20}$$

In paper [22] authors have investigated the case of differential calculus with $\alpha = 1$.

Now we find, that the algebra of 1–forms (3.17) and the algebra of vector fields can be decomposed on the algebra $SL_q(2,C)$ and $U(1)$ subalgebra only for unique value of the parameters

$$\alpha = \frac{2}{1+q^2}, \qquad \beta = \frac{2q^2}{1+q^2}. \tag{3.21}$$

In this case the commutation relations between the parameters and 1–forms are diagonalized after the choice of the new basis of 1–forms

$$\tilde{\theta}^1 = \frac{2}{q+1/q}(q\theta^1 + \frac{1}{q}\theta^4) = \mathrm{Tr}_q \theta$$

$$\tilde{\theta}^4 = \frac{1}{1+q^2}(\theta^1 - \theta^4)$$

$$[\tilde{\theta}^1, a] = [\tilde{\theta}^1, d] = 0 \tag{3.22}$$

$$\begin{array}{ll} \tilde{\theta}^4 a = q^{-2} a\tilde{\theta}^4 & \tilde{\theta}^4 d = q^2 d\tilde{\theta}^4, \\ \theta^2 a = q^{-1} a\theta^2 & \theta^2 d = qd\theta^2, \\ \theta^3 a = q^{-1} a\theta^3 & \theta^3 d = qd\theta^3 \end{array}$$

The other relations are found by making the interchanges $a \leftrightarrow c, d \leftrightarrow b$.

The commutation relations (3.17) can be written as

$$\begin{array}{ll} \tilde{\theta}^1\theta^2 + \theta^2\tilde{\theta}^1 = 0, & \tilde{\theta}^1\theta^3 + \theta^3\tilde{\theta}^1 = 0 \\ q^2\theta^2\tilde{\theta}^4 + q^{-2}\tilde{\theta}^4\theta^2 = 0, & q^2\tilde{\theta}^4\theta^3 + q^{-2}\theta^3\tilde{\theta}^4 = 0 \\ \tilde{\theta}^1\tilde{\theta}^4 + \tilde{\theta}^4\tilde{\theta}^1 = 0, & \theta^2\theta^3 + q^{-2}\theta^3\theta^2 = 0 \end{array} \tag{3.23}$$

$$(\tilde{\theta}^1)^2 = (\theta^2)^2 = (\theta^3)^2 = (\tilde{\theta}^4)^2 = 0$$

For the consistency of the differential algebra of the 1-forms we should define how to order lexicographycally any cubic monomial (Diamond Lemma[7]). For example, if we try to express the cubic monomial $\tilde{\theta}^4\theta^3\theta^2$ of the permutations $(432 \to 423 \to 243 \to 234)$ and of the permutations $(432 \to 342 \to 324 \to 234)$, so we should receive the same result

$$\tilde{\theta}^4\theta^3\theta^2 \to -q^{-4}\theta^3\tilde{\theta}^4\theta^2 \to \theta^3\theta^2\tilde{\theta}^4 \to -q^2\theta^2\theta^3\tilde{\theta}^4;$$

$$\tilde{\theta}^4\theta^3\theta^2 \to -q^2\tilde{\theta}^4\theta^2\theta^3 \to q^6\theta^2\tilde{\theta}^4\theta^3 \to -q^2\theta^2\theta^3\tilde{\theta}^4$$

An attempt to order the monomials $\tilde{\theta}^4\theta^2\tilde{\theta}^1$ and $\tilde{\theta}^4\theta^3\tilde{\theta}^1$ using (3.19–3.20) leads to

$$\left\{ \begin{array}{l} \tilde{\theta}^4\theta^2\tilde{\theta}^1 \to -\tilde{\theta}^4\tilde{\theta}^1\theta^2 \to \tilde{\theta}^1\tilde{\theta}^4\theta^2 \to -q^4\tilde{\theta}^1\theta^2\tilde{\theta}^4 \\ \tilde{\theta}^4\theta^2\tilde{\theta}^1 \to -q^4\theta^2\tilde{\theta}^4\tilde{\theta}^1 \to q^4\theta^2\tilde{\theta}^1\tilde{\theta}^4 \to -q^4\tilde{\theta}^1\theta^2\tilde{\theta}^4 \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \tilde{\theta}^4\theta^3\tilde{\theta}^1 \to -\tilde{\theta}^4\tilde{\theta}^1\theta^3 \to \tilde{\theta}^1\tilde{\theta}^4\theta^3 \to -q^{-4}\tilde{\theta}^1\theta^3\tilde{\theta}^4 \\ \tilde{\theta}^4\theta^3\tilde{\theta}^1 \to -q^{-4}\theta^3\tilde{\theta}^4\tilde{\theta}^1 \to q^{-4}\theta^3\tilde{\theta}^1\tilde{\theta}^4 \to -q^{-4}\tilde{\theta}^1\theta^3\tilde{\theta}^4 \end{array} \right.$$

Normal ordering of monomials like $\tilde{\theta}^4\theta^3\tilde{\theta}^4$, $\theta^2\tilde{\theta}^4\theta^2$, $\theta^3\tilde{\theta}^1\theta^3$, $\tilde{\theta}^1\theta^2\tilde{\theta}^1$ and so on are easily shown to vanish.

Thus, the different ways of ordering cubic monomial lead to the same result and it guarantees the consistency of the differential algebra $GL_q(2, C)$.

In conclusion of this section we find the commutation relations between the parameters of the quantum group and its differentials.

For any parameters of the group K and N we may decompose the expression $K\delta N$ in the complete basis of 1–forms

$$\begin{aligned} K\delta N &= \tilde{A}\delta a + \tilde{B}\delta b + \tilde{C}\delta c + \tilde{D}\delta d = \\ &= (\tilde{A}a + \tilde{C}c)\theta^1 + (\tilde{B}a + \tilde{D}c)\theta^2 + (\tilde{A}b + \tilde{C}d)\theta^3 + (\tilde{B}b + \tilde{D}d)\theta^4 \end{aligned}$$

where $\tilde{A}$,$\tilde{B}$,$\tilde{C}$,$\tilde{D}$ are arbitrary functions of parameters of quantum group.

Using the commutation relations between the parameters and 1–forms we obtain the commutation relations between the parameters and its differentials.

$$\begin{array}{l}
\delta_L a\ a = q^{-2}\ a\ \delta_L a + \frac{q^2-1}{2q^2}\ a^2\ \mathrm{Tr}_q\theta, \\
\delta_L c\ c = q^{-2}\ c\ \delta_L c + \frac{q^2-1}{2q^2}\ c^2\ \mathrm{Tr}_q\theta, \\
\delta_L a\ c = q^{-1}\ c\ \delta_L a + \frac{q^2-1}{2q^2}\ a\ c\ \mathrm{Tr}_q\theta, \\
\delta_L c\ a = q^{-1}\ a\ \delta_L c + (q^{-2}-1)\ c\ \delta_L a + \frac{q^2-1}{2q^2}\ c\ a\ \mathrm{Tr}_q\theta, \\
\delta_L b\ b = q^{2}\ b\ \delta_L b + \frac{1-q^2}{2}\ b^2\ \mathrm{Tr}_q\theta, \\
\delta_L d\ d = q^{2}\ d\ \delta_L d + \frac{1-q^2}{2}\ d^2\ \mathrm{Tr}_q\theta, \\
\delta_L b\ d = q\ d\ \delta_L b + (q^2-1)\ b\ \delta_L d + \frac{1-q^2}{2}\ b\ d\ \mathrm{Tr}_q\theta, \\
\delta_L d\ b = q\ b\ \delta_L d + \frac{1-q^2}{2}\ d\ b\ \mathrm{Tr}_q\theta,
\end{array}$$

$$\begin{array}{l}
\delta_L a\ b = q\ b\ \delta_L a + \frac{(q^2-1)}{q^2\mathcal{D}_q}\ a\ b\ (q\ c\ \delta_L b - a\ \delta_L d) + \frac{q^2-1}{2q^2}\ a\ b\ \mathrm{Tr}_q\theta, \\
\delta_L a\ d = d\ \delta_L a + \lambda\ b\ \delta_L c + \frac{q^2-1}{\mathcal{D}_q}\ a\ d(d\ \delta_L a - \frac{1}{q}\ b\ \delta_L c) - \frac{q^2-1}{2}\ a\ d\ \mathrm{Tr}_q\theta \\
\delta_L c\ b = b\ \delta_L c + \frac{(q^2-1)}{\mathcal{D}_q}\ c\ b\ (d\ \delta_L a - \frac{1}{q}\ b\ \delta_L c) - \frac{(q^2-1)}{2}\ c\ b\ \mathrm{Tr}_q\theta, \\
\delta_L c\ d = q\ d\ \delta_L c + \frac{(q^2-1)}{\mathcal{D}_q}\ c\ d\ (d\ \delta_L a - \frac{1}{q}\ b\ \delta_L c) - \frac{(q^2-1)}{2}\ c\ d\ \mathrm{Tr}_q\theta, \\
\delta_L b\ a = q^{-1}\ a\ \delta_L b + \frac{(q^2-1)}{q^2\mathcal{D}_q}\ b\ a\ (q\ c\ \delta_L b - a\ \delta_L d) + \frac{(q^2-1)}{2q^2}\ b\ a\ \mathrm{Tr}_q\theta, \\
\delta_L b\ c = c\ \delta_L b + \frac{(q^2-1)}{\mathcal{D}_q}\ b\ c\ (d\ \delta_L a - \frac{1}{q}\ b\ \delta_L c) - \frac{(q^2-1)}{2}\ b\ c\ \mathrm{Tr}_q\theta, \\
\delta_L d\ a = a\ \delta_L d - \lambda\ c\ \delta_L b + \frac{q^2-1}{\mathcal{D}_q}\ d\ a\ (d\ \delta_L a - \frac{1}{q}\ b\ \delta_L c) - \frac{(q^2-1)}{2}\ d\ a\ \mathrm{Tr}_q\theta \\
\delta_L d\ c = q^{-1}\ c\ \delta_L d + \frac{(q^2-1)}{\mathcal{D}_q}\ d\ c\ (d\ \delta_L a - \frac{1}{q}\ b\ \delta_L c) - \frac{(q^2-1)}{2}\ d\ c\ \mathrm{Tr}_q\theta,
\end{array} \tag{3.24}$$

where $\mathrm{Tr}_q\theta = \frac{2}{q+1/q}(q\theta^1 + \frac{1}{q}\theta^4) = \tilde{\theta}^1$.

The algebra of vector fields in this case has the following form

$$\begin{array}{c}
\overleftarrow{\nabla}_3\overleftarrow{\nabla}_2 - q^2\ \overleftarrow{\nabla}_2\overleftarrow{\nabla}_3 = \hat{\overleftarrow{\nabla}}_1 \\
q^2\ \overleftarrow{\nabla}_2\ \hat{\overleftarrow{\nabla}}_1 - q^{-2}\hat{\overleftarrow{\nabla}}_1\ \overleftarrow{\nabla}_2 = (1+q^2)\ \overleftarrow{\nabla}_2 \\
q^2\hat{\overleftarrow{\nabla}}_1\ \overleftarrow{\nabla}_3 - q^{-2}\ \overleftarrow{\nabla}_3\ \hat{\overleftarrow{\nabla}}_1 = (1+q^2)\ \overleftarrow{\nabla}_3 \\
[\hat{\overleftarrow{\nabla}}_4, \hat{\overleftarrow{\nabla}}_1] = [\hat{\overleftarrow{\nabla}}_4, \overleftarrow{\nabla}_2] = [\hat{\overleftarrow{\nabla}}_4, \overleftarrow{\nabla}_3] = 0
\end{array} \tag{3.25}$$

where

$$\hat{\overleftarrow{\nabla}}_1 = \overleftarrow{\nabla}_1 - q^2\ \overleftarrow{\nabla}_4, \qquad \hat{\overleftarrow{\nabla}}_4 = \overleftarrow{\nabla}_1 + \overleftarrow{\nabla}_4 \tag{3.26}$$

After the mapping $\hat{\overleftarrow{\nabla}}_1, \overleftarrow{\nabla}_2, \overleftarrow{\nabla}_3, \hat{\overleftarrow{\nabla}}_4$ to the new generators H, T_2, T_3, N

$$\hat{\overleftarrow{\nabla}}_1 = \frac{1-q^{-2H}}{1-q^{-2}} \quad , \quad \overleftarrow{\nabla}_2 = q^{-H/2}T_2,$$

$$\hat{\overleftarrow{\nabla}}_4 = N \qquad , \qquad \overleftarrow{\nabla}_3 = q^{-H/2}T_3, \tag{3.27}$$

we obtain the algebra $U_qGL(2,C)$ in the form of a Drinfeld–Jimbo algebra [1,2]

$$[T_3, T_2] = \frac{q^H - q^{-H}}{q - q^{-1}}$$

$$[H,T_3]=2T_3 \quad , \quad [H,T_2]=-2T_2 \tag{3.28}$$
$$[N,H]=[N,T_2]=[N,T_3]=0$$

Note, that there exists one more matched solution when $\alpha\to 0$ and $\beta\to 0$, but its ratio remains equal $\frac{\beta}{\alpha}=q^2$. In this case we have from (3.12)

$$\begin{array}{rclrcl}
(\theta^1+\theta^4)a &=& \frac{1}{q^2}a(\theta^1+\theta^4) & \theta^1 d &=& -d\,\theta^4\\
(\theta^1+q^2\theta^4)a &=& a(\theta^1+q^2\theta^4) & \theta^4\, d &=& (1+q^2)d\theta^4+q^2d\theta^1\\
\theta^2 a &=& \frac{1}{q}a\theta^2 & \theta^1 a &=& (1+q^{-2})a\theta^1+q^{-2}a\theta^4\\
\theta^3 a &=& \frac{1}{q}a\theta^3 & \theta^4 a &=& -a\theta^1\\
\\
(\theta^1+\theta^4)d &=& q^2d(\theta^1+\theta^4) & & &\\
(\theta^1+q^2\theta^4)d &=& d(\theta^1+q^2\theta^4) & & &\\
\theta^2 d &=& qd\theta^2 & & &\\
\theta^3 d &=& qd\theta^3 & & &
\end{array} \tag{3.36}$$

The other relations are found by making interchanges $d\to b$, $a\to c$. The commutation relations between θ^2,θ^3 and a,b,c,d did not change.

4 Left differential calculus on $SL_q(2,C)$ and quantum plane.

To obtain the differential calculus on the $SL_q(2,C)$ it is necessary to suppose additional constraints $\mathcal{D}_q=1$ and $\delta\mathcal{D}_q=\mathcal{D}_q\mathrm{Tr}_q=0$. In an opposite way to the bicovariant calculus it is possible, so the quantum determinant $\mathcal{D}_q$ commutes with 1–forms and $\delta\mathcal{D}_q=0$ satisfies by vanishing $\tilde{\theta}^1=\mathrm{Tr}_q\theta=\alpha(q\theta^1+\frac{1}{q}\theta^4)=0$. As a consequence of these conditions we obtained three–dimensional differential calculus, which is independent of the parameters α and β.

Thus we have next commutation relations in the $SL_q(2,C)$ case

$$\begin{array}{lcl}
\theta^1 a=q^{-2}a\theta^1 & & \theta^1 d=q^2 d\theta^1\\
\theta^2 a=q^{-1}a\theta^2 & , & \theta^2 d=qd\theta^2\\
\theta^3 a=q^{-1}a\theta^3 & & \theta^3 d=qd\theta^3
\end{array} \tag{4.1}$$

$$\begin{array}{l}
(\theta^1)^2=(\theta^2)^2=(\theta^3)^2=0, \qquad \theta^4=-q^2\theta^1\\
\theta^1\theta^2+q^4\theta^2\theta^1=0\\
\theta^1\theta^3+q^{-4}\theta^3\theta^1=0\\
\theta^2\theta^3+q^{-2}\theta^3\theta^2=0
\end{array} \tag{4.2}$$

and other relations are made by interchanges $a\leftrightarrow c$, $d\leftrightarrow b$.

Notice, that similary commutation relations for $SU_q(2)$ were received by Woronowich [1]. The algebra of vector fields in this case has the following form

$$\begin{aligned} q^2\nabla_1\nabla_3 - q^{-2}\nabla_3\nabla_1 &= (1+q^2)\nabla_3 \\ q^2\nabla_2\nabla_1 - q^{-2}\nabla_1\nabla_2 &= (1+q^2)\nabla_2 \\ \nabla_3\nabla_2 - q^2\nabla_2\nabla_3 &= \nabla_1 \end{aligned} \tag{4.3}$$

The commutation relations between the parameters and its differentials have the form:

$$\begin{aligned} \delta_L a\, a &= q^{-2} a\, \delta_L a \\ \delta_L c\, c &= q^{-2} c\, \delta_L c \\ \delta_L a\, c &= q^{-1} c\, \delta_L a \\ \delta_L c\, a &= q^{-1} a\, \delta_L c + (q^{-2}-1) c\, \delta_L a \end{aligned} \quad , \quad \begin{aligned} \delta_L b\, b &= q^2 b\, \delta_L b \\ \delta_L d\, d &= q^2 d\, \delta_L d \\ \delta_L b\, d &= q\, d\, \delta_L b + (q^2-1) b\, \delta_L d \\ \delta_L d\, b &= q\, b\, \delta_L d \end{aligned}$$

$$\begin{aligned} \delta_L a\, b &= q\, b\, \delta_L a + (q^2-1) a\, b\, d\, \delta_L a + (\tfrac{1}{q} - q) a\, b^2 \delta_L c \\ \delta_L a\, d &= q^2 d\, \delta_L a + q(q^2-1) b\, c\, d\, \delta_L a + (1-q^2) b^2 c\, \delta_L c \\ \delta_L c\, b &= b\, \delta_L c + (q^2-1) b\, c\, d\, \delta_L a + (q^{-1}-q) b^2 c\, \delta_L c \\ \delta_L c\, d &= q\, d\, \delta_L c + (q^2-1) c\, d^2 \delta_L a + (q^{-1}-q) c\, d\, b\, \delta_L c \\ \delta_L b\, a &= q^{-1} a\, \delta_L b + (q^2-1) b\, a\, d\, \delta_L a + (1-q^2) b^2\, a\, \delta_L c \\ \delta_L b\, c &= c\, \delta_L b + (q^2-1) b\, c\, d\, \delta_L a + (q^{-1}-q) b^2 c\, \delta_L c \\ \delta_L d\, a &= q^{-2} a\, \delta_L d + q^{-2}(q^{-1}-q) b\, c\, a\, \delta_L d + (1-q^{-2}) b\, c^2 \delta_L b \\ \delta_L d\, c &= q^{-1} c\, \delta_L d + (q^{-2}-1) d\, c\, a\, \delta_L d + (q-q^{-1}) d\, c^2 \delta_L b \end{aligned} \tag{4.4}$$

Let us mention that for the differential calculus described in (4.4) there are the quartic powers of the group elements essentially in contrast to the commutation relations between T^i_k and the Maurer–Cartan left invariant 1–forms. The quartic powers in the commutation relations have been obtained in papers [13,14].

Another difference, contrary to the bicovariant R-matrix approach of Manin[7], Demidov[16], Castellani, Ashieri[17] appears. Let us call to mind the differential calculus on the quantum plane $C_q(2|0)$ of Wess–Zumino[6]:

$$\text{I)}\quad \begin{aligned} &xy = q\, yx \\ &\delta x\, x = q^{-2} x\, \delta x \\ &\delta y\, y = q^2 y\, \delta y \\ &\delta x\, y = q\, y\, \delta x + (q^2-1) x\, \delta y \\ &\delta y\, x = q\, x\, \delta y \end{aligned} \qquad \text{II)}\quad \begin{aligned} &q \to q^{-1} \quad x \to y \quad y \to x \\ &\delta x\, y = q^{-1}\, y\, \delta x \\ &\delta y\, x = q^{-1}\, x\, \delta y + (q^{-2}-1)\, y\, \delta x \end{aligned} \tag{4.5}$$

It is possible to introduce a quantum plane $C_q(2|0)$ as the Hopf algebra surjection $\pi : SL_q(2,C) \to C_q(2|0)$ such that $\pi(T^1_2) = \pi(b) = 0$ or $\pi(T^2_1) = \pi(c) = 0$:

$$\mathcal{D}_q = \det{}_q T = ad = 1$$

$$T_- = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix};$$

$$T_+ = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} y^{-1} & x \\ 0 & y \end{pmatrix};$$

We see, that in the case $\pi(c) = 0$ or $\pi(b) = 0$ commutation relations (4.4) form the commutation relations on the quantum plane.

The algebra of left 1–forms and the algebra of vector fields can be obtained if we put in (4.2)

$$\begin{array}{l} 1) \quad c = 0, \quad \theta^3 = 0, \quad \nabla_3 = 0 \\ 2) \quad b = 0, \quad \theta^2 = 0, \quad \nabla_2 = 0 \end{array}$$

Hence, we received the differential calculus on the quantum group $GL_q(2,C)$ and its subgroup with the property of natural reduction ("quantum matrjoshka"): the differential calculus on the $GL_q(2,C)$ contains the differential calculus on the $SL_q(2,C)$, which also contains the differential calculus on the quantum plane $C_q(2|0)$.

5 Right differential calculus on $GL_q(2,C)$.

In this section we give a construction of the right differential calculi on $GL_q(2,C)$.

Analogously to previous sections we will assume, that the commutation relations between the right 1-forms ω and parameters a,b,c,d are given by

$$T^k_n \omega^\alpha_\beta = \omega^\delta_\gamma \left(Q^k_m\right)^{\gamma\alpha}_{\beta\delta} T^m_n, \quad \text{where } Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{5.1}$$

or

$$a\omega^k = \omega^l A^k_l a + \omega^l B^k_l c, \quad c\omega^k = \omega^l D^k_l c + \omega^l C^k_l a,$$
$$b\omega^k = \omega^l A^k_l b + \omega^l B^k_l d, \quad d\omega^k = \omega^l D^k_l d + \omega^l C^k_l b,$$

By using the commutation relations given above it is now easy to prove, that

$$(R_{12}T_1T_2 - T_2T_1R_{12})\omega^k = 0$$

leads to the equations

$$R_{12}Q_1Q_2 = Q_2Q_1R_{12} \tag{5.2}$$

It is an important property, that the quantum determinant be a central for ω^k also

$$\mathcal{D}_q\omega^k = (AD - qBC)\omega^k\mathcal{D}_q \tag{5.3}$$

and we have

$$AD - qBC = 1 \tag{5.4}$$

Acting by the right exterior differential $\overrightarrow{\delta_R}$ on $(RTT - TTR)$ relation and on the different expressions for the quantum determinant $\mathcal{D}_q$

$$\begin{aligned}
\overrightarrow{\delta_R}\,(ad - qbc) &= (\omega^1 + \omega^k A_k^4 - q\omega^k B_k^3)\mathcal{D}_q \\
\overrightarrow{\delta_R}\,(ad - qbc) &= (\omega^1 + \omega^k A_k^4 - q\omega^k C_k^2)\mathcal{D}_q \\
\overrightarrow{\delta_R}\,(da - q^{-1}bc) &= (\omega^4 + \omega^k D_k^1 - q{-}1\omega^k B_k^3)\mathcal{D}_q \\
\overrightarrow{\delta_R}\,(da - q^{-1}bc) &= (\omega^4 + \omega^k D_k^1 - q{-}1\omega^k C_k^2)\mathcal{D}_q
\end{aligned} \tag{5.5}$$

we obtain the following consistency conditions

$$\begin{gathered}
C_k^2 = B_k^3 \\
\omega^3 + \omega^k C_k^1 - q^{-1}\omega^k A_k^3 = 0 \\
\omega^2 + \omega^k B_k^4 - q\omega^k D_k^2 = 0
\end{gathered} \tag{5.6}$$

By analogy with previous section we choose the representation of Q, where

$$B = C = 0, \quad AD = 1 \tag{5.7}$$

In this case we obtain the following anzatz for the matrices A and D

$$A = \begin{pmatrix} 1 - t + \frac{t}{u} & 0 & 0 & (1-t)\frac{t}{u} \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ (u-1) & 0 & 0 & t \end{pmatrix}, \quad D = \begin{pmatrix} u & 0 & 0 & t-1 \\ 0 & 1/q & 0 & 0 \\ 0 & 0 & q & 0 \\ (1-u)\frac{u}{t} & 0 & 0 & (1 - u + \frac{u}{t}) \end{pmatrix}, \tag{5.8}$$

where $t(q)$ and $u(q)$ are some functions of q.

Using these matrices and applying right exterior differential $\overrightarrow{\delta_R}$ to (5.1) together with the Cartan–Maurer equations for the right–invariant 1-forms ω^k $\overrightarrow{\delta_R}\,\omega = \omega^2$ we can obtain the following algebra of ω-forms

$$\begin{gathered}
(\omega^1)^2 = (\omega^2)^2 = (\omega^3)^2 = (\omega^4)^2 = 0 \\
\omega^3\omega^2 + q^{-2}\omega^2\omega^3 = 0 \\
\omega^1\omega^4 + \omega^4\omega^1 = 0 \\
u\omega^1\omega^3 + (1 - t + \frac{t}{u})\omega^3\omega^1 = (t-1)(q^2\omega^3\omega^4 - \omega^4\omega^3), \\
(u-1)(q^{-2}\omega^1\omega^3 - \omega^3\omega^1) = q^2 u\omega^3\omega^4 + q^{-2}(1 - t + \frac{t}{u})\omega^4\omega^3, \\
u\omega^2\omega^1 + (1 - t + \frac{t}{u})\omega^1\omega^2 = (t-1)(q^2\omega^4\omega^2 - \omega^2\omega^4), \\
(u-1)(q^{-2}\omega^2\omega^1 - \omega^1\omega^2) = q^2 u\omega^4\omega^2 + q^{-2}(1 - t + \frac{t}{u})\omega^2\omega^4
\end{gathered} \tag{5.9}$$

and the condition

$$t = u\, q^2 \tag{5.10}$$

Hence, taking into account (5.8), we get for (5.5)

$$\vec{\delta}_R\, \mathcal{D}_q = t(q^{-2}\omega^1 + \omega^4)\mathcal{D}_q \tag{5.11}$$

The parameter t holds fixed form for the $\mathrm{Tr}_q\omega$.

By definition, the effect of applying right differential to an arbitrary function f on the quantum group is

$$\vec{\delta}_R\, f = \vec{\delta}_R\, T^i_k \frac{\vec{\partial}}{\partial T^i_k} f = \omega^i\, \vec{\nabla}_i\, f \tag{5.12}$$

where $\vec{\nabla}_i$ are the right vector fields on the quantum group.

From the condition

$$\vec{\delta}_R^{\,2}\; f = 0 \tag{5.13}$$

and taking into account (5.9) we find a right vector field algebra of arbitrary t.

$$q^{-2}\, \vec{\nabla}_2\, \hat{\vec{\nabla}}_1 - q^2 \hat{\vec{\nabla}}_1\, \vec{\nabla}_2 = (1+q^{-2})\, \vec{\nabla}_2,$$

$$q^{-2}\hat{\vec{\nabla}}_1\, \vec{\nabla}_3 - q^2\, \vec{\nabla}_3\, \hat{\vec{\nabla}}_1 = (1+q^{-2})\, \vec{\nabla}_3, \tag{5.14}$$

$$\vec{\nabla}_3\vec{\nabla}_2 - q^{-2}\, \vec{\nabla}_2\vec{\nabla}_3 = \hat{\vec{\nabla}}_1,$$

$$\begin{aligned} [\vec{\nabla}_4, \vec{\nabla}_2] &= q^2(t-1)(q^2+1)\hat{\vec{\nabla}}_1\, \vec{\nabla}_2 + q^2(t+q^{-2}t-1)\, \vec{\nabla}_2, \\ [\vec{\nabla}_3, \vec{\nabla}_4] &= q^2(t-1)(q^2+1)\, \vec{\nabla}_3\, \hat{\vec{\nabla}}_1 + q^2(t+q^{-2}t-1)\, \vec{\nabla}_3, \end{aligned} \tag{5.15}$$

where

$$\hat{\vec{\nabla}}_1 = \vec{\nabla}_1 - q^{-2}\, \vec{\nabla}_4 \tag{5.16}$$

Again we have the decomposition of these commutation relations on the $SL_q(2,C)$ and $U(1)$ subalgebras only for

$$t = \frac{2}{1+q^{-2}}, \quad u = \frac{2}{1+q^2} \tag{5.17}$$

Commutation relations (5.14) form the algebra for $SL_q(2,C)$ and for $U(1)$ we have

$$[\hat{\vec{\nabla}}_4, \hat{\vec{\nabla}}_1] = [\hat{\vec{\nabla}}_4, \vec{\nabla}_2] = [\hat{\vec{\nabla}}_4, \vec{\nabla}_3] = 0 \tag{5.18}$$

where $\hat{\vec{\nabla}}_4 = \vec{\nabla}_1 + \vec{\nabla}_4$.

After the mapping from $\hat{\vec{\nabla}}_1, \vec{\nabla}_2, \vec{\nabla}_3, \hat{\vec{\nabla}}_4$ to new generators H, T_2, T_3, N

$$\hat{\vec{\nabla}}_1 = \frac{1-q^{2H}}{1-q^2}, \quad \vec{\nabla}_2 = q^{H/2}T_2, \quad \vec{\nabla}_3 = q^{H/2}T_3, \quad \hat{\vec{\nabla}}_4 = N$$

we again obtain the algebra $U_qGL(2,C)$ in the form of the Drinfeld–Jimbo algebra (3.34).

Writing ω^1 and ω^4 as linear combinations

$$\bar{\omega}^1 = \frac{2}{q+q^{-1}}(\frac{1}{q}\omega^1 + q\omega^4) = \mathrm{Tr}_q\omega, \tag{5.19}$$

$$\bar{\omega}^4 = \frac{1}{1+q^{-2}}(\omega^1 - \omega^4)$$

we obtain more simple commutation relations for right 1–forms ω

$$\begin{gathered}
[a, \bar{\omega}^1] = [d, \bar{\omega}^1] = 0 \\
\begin{aligned}
a\bar{\omega}^4 &= q^2\bar{\omega}^4 a, & \qquad d\bar{\omega}^4 &= q^{-2}\bar{\omega}^4 d \\
a\omega^2 &= q\omega^2 a, & \qquad d\omega^2 &= q^{-1}\omega^2 d \\
a\omega^3 &= q\omega^3 a, & \qquad d\omega^3 &= q^{-1}\omega^3 d
\end{aligned}
\end{gathered} \tag{5.20}$$

The other relations are found by making the interchanges $a \to b$, $d \to c$.

The commutation relations between ω have the following form

$$\bar{\omega}^1\omega^2 + \omega^2\bar{\omega}^1 = 0, \quad \bar{\omega}^1\omega^3 + \omega^3\bar{\omega}^1 = 0, \quad \bar{\omega}^1\bar{\omega}^4 + \bar{\omega}^4\bar{\omega}^1 = 0 \tag{5.21}$$

$$q^2\omega^3\bar{\omega}^4 + q^{-2}\bar{\omega}^4\omega^3 = 0, \quad q^{-2}\omega^2\bar{\omega}^4 + q^2\bar{\omega}^4\omega^2 = 0$$

Thus we recovered the results [10] for right 1–forms.

Note, that commutation relations for right 1–forms ω can be obtained from commutation relations for left 1–forms θ by making interchanges ($a \leftrightarrow d$,$q \leftrightarrow 1/q$, $\theta \leftrightarrow \omega$).

In the terms of the right differentials commutation relations between the parameters

and its differential have the following form for the $GL_q(2,C)$

$$
\begin{aligned}
&a\ \overrightarrow{\delta}_R\ a = q^2\ \ \overrightarrow{\delta}_R\ a\ a + \tfrac{1-q^2}{2}\mathrm{Tr}_q\omega\ a^2\\
&b\ \overrightarrow{\delta}_R\ b = q^2\ \ \overrightarrow{\delta}_R\ b\ b + \tfrac{1-q^2}{2}\mathrm{Tr}_q\omega\ b^2\\
&c\ \overrightarrow{\delta}_R\ c = q^{-2}\ \ \overrightarrow{\delta}_R\ b\ b + \tfrac{1-q^{-2}}{2}\mathrm{Tr}_q\omega\ c^2\\
&d\ \overrightarrow{\delta}_R\ a = q^{-2}\ \ \overrightarrow{\delta}_R\ d\ d + \tfrac{1-q^{-2}}{2}\mathrm{Tr}_q\omega\ d^2\\
&b\ \overrightarrow{\delta}_R\ a = q\ \ \overrightarrow{\delta}_R\ a\ b + \tfrac{1-q^2}{2}\mathrm{Tr}_q\omega\ b\ a\\
&a\ \overrightarrow{\delta}_R\ b = q\ \ \overrightarrow{\delta}_R\ b\ a + (q^2-1)\ \overrightarrow{\delta}_R\ a\ b + \tfrac{1-q^2}{2}\mathrm{Tr}_q\omega\ a\ b\\
&d\ \overrightarrow{\delta}_R\ c = \tfrac{1}{q}\ \overrightarrow{\delta}_R\ c\ d + (q^{-2}-1)\ \overrightarrow{\delta}_R\ d\ c + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ d\ c\\
&c\ \overrightarrow{\delta}_R\ d = \tfrac{1}{q}\ \overrightarrow{\delta}_R\ d\ c + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ c\ d\\
&a\ \overrightarrow{\delta}_R\ c = q\ \overrightarrow{\delta}_R\ c\ a + \tfrac{(q-1/q)}{\mathcal{D}_q}(\overrightarrow{\delta}_R\ b\ c - q^{-1}\ \overrightarrow{\delta}_R\ a\ d)\ a\ c + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ a\ c\\
&c\ \overrightarrow{\delta}_R\ a = \tfrac{1}{q}\ \overrightarrow{\delta}_R\ a\ c + \tfrac{(q-1/q)}{\mathcal{D}_q}(\overrightarrow{\delta}_R\ b\ c - q^{-1}\ \overrightarrow{\delta}_R\ a\ d)\ c\ a + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ c\ a\\
&a\ \overrightarrow{\delta}_R\ d = \overrightarrow{\delta}_R\ d\ a + \tfrac{(q-1/q)}{\mathcal{D}_q}(\overrightarrow{\delta}_R\ b\ c - q^{-1}\ \overrightarrow{\delta}_R\ a\ d)\ a\ d + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ a\ d\\
&d\ \overrightarrow{\delta}_R\ a = \overrightarrow{\delta}_R\ a\ d + \tfrac{(q-1/q)}{\mathcal{D}_q}(\overrightarrow{\delta}_R\ b\ c - q^{-1}\ \overrightarrow{\delta}_R\ a\ d)\ d\ a + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ d\ a\\
&b\ \overrightarrow{\delta}_R\ c = \overrightarrow{\delta}_R\ c\ b + \tfrac{(q-1/q)}{\mathcal{D}_q}(\overrightarrow{\delta}_R\ b\ c - q^{-1}\ \overrightarrow{\delta}_R\ a\ d)\ b\ c + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ b\ c\\
&c\ \overrightarrow{\delta}_R\ b = \overrightarrow{\delta}_R\ b\ c + \tfrac{(q-1/q)}{\mathcal{D}_q}(\overrightarrow{\delta}_R\ b\ c - q^{-1}\ \overrightarrow{\delta}_R\ a\ d)\ c\ b + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ c\ b\\
&b\ \overrightarrow{\delta}_R\ d = \overrightarrow{\delta}_R\ d\ b + \tfrac{(q-1/q)}{\mathcal{D}_q}(\overrightarrow{\delta}_R\ b\ c - q^{-1}\ \overrightarrow{\delta}_R\ a\ d)\ b\ d + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ b\ d\\
&d\ \overrightarrow{\delta}_R\ b = \overrightarrow{\delta}_R\ b\ d + \tfrac{(q-1/q)}{\mathcal{D}_q}(\overrightarrow{\delta}_R\ b\ c - q^{-1}\ \overrightarrow{\delta}_R\ a\ d)\ d\ b + \tfrac{(1-q^{-2})}{2}\mathrm{Tr}_q\omega\ d\ b
\end{aligned}
\tag{5.22}
$$

Again the choice $\mathcal{D}_q = 1,\ \ \delta\mathcal{D}_q = \bar{\omega}^1\mathcal{D}_q = 0$

$$
\bar{\omega}^1 = \mathrm{Tr}_q\omega = \frac{2}{q+1/q}(\frac{1}{q}\omega^1 + q\omega^4) = 0 \tag{5.23}
$$

$$
\bar{\omega}^4 = \omega^1
$$

leads to the commutation relations for the $SL_q(2,C)$. It is not hard to see that (5.35) with the conditions (5.36) have the form of the solutions of Wess–Zumino for the quantum plane $C_q(2|0)$.

It will be noticed that now another combination of the parameters, namely

$$
ab = q\ ba, \qquad cd = q\ dc
$$

and its differentials are quantum planes.

If we apply $\pi(b) = 0$

$$
T_- = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} y^{-1} & 0 \\ x & y \end{pmatrix}
$$

and $\pi(c) = 0$

$$
T_+ = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}
$$

so (5.22) corresponds to the solution of Wess–Zumino for the quantum plane $C_q(2|0)$. Thus we see, that the right differential calculus prefers the rows, but the left differential calculus prefers the columns. At last we show the difference between matched differential calculus and bicovariant differential calculus for $SL_q(2,C)$ group. Using commutation relations (5.20) for right forms ω^k, we can obtain commutation relations for left forms θ^k after transformation $\theta = g^{-1}\omega g$. For bicovariant differential calculus this commutation relation must be the same as for left differential calculus. For matched differential calculus we obtained commutation relations between the left 1–forms θ^k and higher degrees monomials, for example

$$\theta^1 b = \frac{1}{q^2} b\theta^1 - \frac{(q^4-1)}{q^4} b^2 acd\theta^1 - \frac{(q^2-1)}{q^3} b^2 ad^2\theta^3 + \frac{(q^2-1)}{q^4} b^2 ac^2\theta^2$$

$$\theta^1 c = q^2 c\theta^1 + q^2(q^4-1)dc^2 ba\theta^1 + q^2(q^2-1)dc^2 b^2\theta^3 - q(q^2-1)dc^2 a^2\theta^2$$

$$\theta^2 b = \frac{1}{q^3} b\theta^2 - \frac{(q^4-1)}{q^5} b^3 cd\theta^1 - \frac{(q^2-1)}{q^4} b^3 d^2\theta^3 + \frac{(q^2-1)}{q^5} b^3 c^2\theta^2$$

$$\theta^2 c = qc\theta^2 + (q^4-1)cd^2 ba\theta^1 + (q^2-1)cd^2 b^2\theta^3 - \frac{(q^2-1)}{q} cd^2 a^2\theta^2$$

This solution is the solution of the equation (3.1) for matched differential calculus also.

6 Conclusions

We proposed the construction of the differential calculus on the quantum group and its subgroup with the property of the natural reduction: the differential calculus on the quantum group $GL_q(2,C)$ has to contain the differential calculus on the quantum subgroup $SL_q(2,C)$ and quantum plane $C_q(2|0)$ ("quantum matrjoshka"). We found, that there are two differential calculi, associated to the left differential Maurer–Cartan 1-forms and to the right differential 1-forms. Matched reduction take the degeneracy between the left and right differentials.

The space of 1-forms is four–dimensional for the quantum group $GL_q(2,C)$ and is three–dimensional for the $SL_q(2,C)$. The quantum determinant $\mathcal{D}_q$ is central element for 1-forms also. The obvious way to carry differential calculus from $GL_q(2,C)$ over to $SL_q(2,C)$ by imposing the determinant constraint $\mathcal{D}_q = 1$ works with constraint $\mathrm{Tr}_q\theta = \frac{2}{q+1/q}(q\theta^1 + \frac{1}{q}\theta^4) = 0$.

Next step to carry differential calculus from $SL_q(2,C)$ over to $C_q(2|0)$ is to impose the constraint $\theta^2 = 0$ or $\theta^3 = 0$, that is equivalent to $\pi(b) = 0$ or $\pi(c) = 0$.

The correspondence between left and right differential calculi is based on well known property of R–matrices

$$R_q = R^{-1}_{1/q}$$

and by making the interchanges

$$(a \to d,\ d \to a,\ q \to 1/q,\ b \to b,\ c \to c)$$

The classical limit ($q \to 1$) of the "left" differential calculus and the "right" differential calculus is the undeformed ordinary differential calculus.

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